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Derivative nonlinear Schrödinger equations and Hermitian symmetric spaces

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Abstract. It is shown that to each Hermitian symmetric space there corresponds an integrable system of generalised derivative nonlinear Schrödinger equations (DNLS). The nonlinear terms are related to the curvature tensor of the associated symmetric space. The Hamiltonian form of these equations is presented. These results are an extension of those presented in an earlier paper on generalised NLS equations associated with symmetric and reductive homogeneous spaces.

1. Introduction

The nonlinear Schrödinger equation (NLS)

$$\mathrm{i}q_t = q_{xx} + 2q^*q^2 \tag{1.1a}$$

and its derivative form (DNLS)

$$\mathrm{i}q_t = q_{xx} + 2\mathrm{i}(q^*q^2)_x \tag{1.1b}$$

have important applications in such fields as plasma physics and nonlinear optics, and are well known to be completely integrable Hamiltonian systems. There exist vector generalisations of each of these equations:

$$\mathrm{i}q_{jt} = q_{jxx} + 2\sum_{k=1}^{n} q_k^* q_k q_j \qquad \mathrm{i}q_{jt} = q_{jxx} + 2\mathrm{i}\left(\sum_{k=1}^{n} q_k^* q_k q_j\right)_x \tag{1.2a, b}$$

which are also soluble by means of the inverse scattering technique. The appropriate linear scattering problems are respectively:

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{pmatrix}_x = \left(\begin{array}{c|ccc} n\lambda & q_1 & \cdots & q_n \\ \hline -q_1^* & -\lambda & & \\ \vdots & & \ddots & \\ -q_n^* & & & -\lambda \end{array}\right) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{pmatrix} \tag{1.3a}$$

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{pmatrix}_x = \left(\begin{array}{c|ccc} n\lambda^2 & \lambda q_1 & \cdots & \lambda q_n \\ \hline -\lambda q_1^* & -\lambda^2 & & \\ \vdots & & \ddots & \\ -\lambda q_n^* & & & -\lambda^2 \end{array}\right) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{pmatrix}. \tag{1.3b}$$

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When $n=1$ these eigenvalue problems correspond to the single component equations (1.1) and were respectively discovered by Zakharov and Shabat (1972) and Kaup and Newell (1978).

It is well known that equations (1.2) are associated with complex projective space $\mathbb{C}P^n \simeq SU(n+1)/(S(U(1)\times U(n)))$, as is evident from the form of the matrices in (1.3). It was recently pointed out (Fordy and Kulish 1983) that the *form* of the multicomponent NLS equation (1.2*a*) is very closely related to the symmetric space $\mathbb{C}P^n$. This symmetric space is Hermitian and it is this feature which was exploited in the above cited paper.

For each Hermitian symmetric space G/K there is a very special element A of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{k}$ of K is given by $\mathfrak{k} = C_{\mathfrak{g}}(A) = \{B \in \mathfrak{g}: [A, B] = 0\}$. In the above paper the linear problem (1.3a) was generalised by:

$$\phi_x = (\lambda A + Q(x,t))\phi \tag{1.4}$$

where $Q(x,t) \in T_{p_0}(G/K) =$ tangent space to G/K at point p_0. The second-order isospectral flow of (1.4) has cubic interaction term and the coupling coefficients are just the components of the Riemann curvature tensor of G/K. This is a direct generalisation of (1.2a), so these equations are generalised NLS equations.

In the present paper the corresponding generalisation of (1.3b) is considered:

$$\phi_x = (\lambda^2 A + \lambda Q(x,t))\phi. \tag{1.5}$$

The second-order flow of (1.5) is now a generalisation of the DNLS equation (1.2*b*). The coupling coefficients are still the components of the Riemann tensor, but this time the interaction term is the derivative of a cubic nonlinearity. These equations have Hamiltonian form (3.13) given in terms of invariant quantities associated with the symmetric space. As with (1.1b), the Hamiltonian structure is not canonical, but is a constant matrix multiple of $\mathrm{d}/\mathrm{d}x$. This matrix is the inverse of the metric tensor at point p_0 of the symmetric space G/K.

Homogeneous and symmetric spaces have often appeared in the literature, mainly in the context of nonlinear sigma models and chiral fields. Some of this literature discusses the existence of Lax pairs for these systems (Zakharov and Mikhailov 1978, Eichenherr and Pohlmeyer 1979, D'Auria *et al* 1980, Eichenherr and Forger 1981). However, these authors have not found a close relationship (such as in (3.12)) between the nonlinearity and the curvature tensor. It is possible that if some of these calculations were reworked, such a relationship could be derived.

Section 2 reviews some mathematical preliminaries concerning Lie algebras and symmetric spaces. Section 3 derives the general results concerning DNLS equations associated with Hermitian symmetric spaces. Examples will be found in § 4.

2. Mathematical preliminaries

In this section we state a number of relevant facts concerning simple Lie algebras and symmetric spaces. Irreducible symmetric spaces are classified in terms of simple Lie algebras, so we have no need of anything more general in this paper. We give the barest of details. The full theory can be found in (Helgason 1978, Humphreys 1972, Kobayashi and Nomizu 1969).

2.1. *Simple Lie algebras; Cartan–Weyl basis*

In terms of the Cartan–Weyl basis a complex, simple Lie algebra $\mathfrak{g}$ has the following commutation relations (Helgason 1978, Humphreys 1972)

(i) $[h_i, h_j] = 0 \qquad \forall h_i, h_j \in \mathfrak{h}$

(ii) $[h, e_\alpha] = \alpha(h) e_\alpha \qquad \forall h \in \mathfrak{h}, \alpha \in \Phi$

(iii) $[e_\gamma, e_{-\gamma}] = h_\gamma = \sum_{i=1}^{l} d_{\gamma i} h_i$

(iv) $[e_\gamma, e_\beta] = \begin{cases} N_{\gamma,\beta} e_{\gamma+\beta} & 0 \neq \gamma+\beta \in \Phi \\ 0 & \gamma+\beta \notin \Phi. \end{cases}$

It will be necessary to explain some of the terms.

(a) $\mathfrak{h}$ is the Cartan subalgebra, which is the maximal Abelian subalgebra of diagonalisable elements of $\mathfrak{g}$. $\mathfrak{h}$ has basis $\{h_i\}_1^l$ and $d_{\gamma i}$ are the components of $[e_\gamma, e_{-\gamma}] \in \mathfrak{h}$ with respect to this basis. The number l is the rank of the algebra.

(b) $\alpha : \mathfrak{h} \to \mathbb{C}$ are linear functionals called roots, on $\mathfrak{h}$ and their values on some $h \in \mathfrak{h}$ are the eigenvalues of the matrix ad h. The corresponding eigenvectors, e_α, are called root vectors. The space of roots is called Φ.

(c) The coefficients $N_{\alpha,\beta}$ are the most complicated part of these commutation relations. However, for the purposes of this paper their values are irrelevant.

2.2. *Hermitian symmetric spaces*

A homogeneous space of a Lie group G is any differentiable manifold M on which G acts transitively ($\forall p_1, p_2 \in M, \exists g \in G / g \cdot p_1 = p_2$). The subgroup of G which leaves a given point $p_0 \in M$ fixed, is called the isotropy group at p_0 and is defined by:

$$K \equiv K_{p_0} = \{g \in G\colon g \cdot p_0 = p_0\}.$$

It is a theorem that each such M can be identified with a coset space G/K for some subgroup K and that this K plays the role of isotropy group of some point. There are many topological and differential geometric subtleties, but we have no need of them in this paper. We are only interested in the decompositions of the corresponding Lie algebras.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively, and let $\mathfrak{m}$ be the vector space complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \tag{2.2}$$

and $\mathfrak{m}$ is identified with the tangent space $T_{p_0}M$ of $M = G/K$ at point p_0. At the moment we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, but know nothing of $[\mathfrak{k}, \mathfrak{m}]$ and $[\mathfrak{m}, \mathfrak{m}]$.

When $\mathfrak{g}$ satisfies the conditions†:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \tag{2.3}$$

then $\mathfrak{g}$ is called a symmetric algebra and G/K is a symmetric space. These spaces have a metric structure which is given by the restriction of the Killing form to $\mathfrak{m}$. As usual, there exists a torsion free connection. Evaluated at fixed point p_0, the curvature

† Helgason demands that $\mathfrak{k}$ be compact. This corresponds to the metric being positive definite.

tensor is given purely in terms of the Lie bracket operation on $\mathfrak{m}$:

$$(R(X, Y)Z)_{p_0} = -[[X, Y], Z], \qquad X, Y, Z \in \mathfrak{m}. \tag{2.4}$$

The components R^i_{jkl} of the curvature tensor with respect to a basis X_i of $T_{p_0}M$ are defined by:

$$R(X_k, X_l)X_j = R^i_{jkl}X_i. \tag{2.5}$$

The corresponding metric is given by the Killing form

$$g(X, Y) = \operatorname{Tr} \operatorname{ad} X \operatorname{ad} Y, \qquad g_{ij} = g(X_i, X_j). \tag{2.6}$$

Tensorial indices are lowered and raised in the usual way by means of the metric tensor and its inverse.

For spaces of constant curvature, the Riemann curvature tensor is related to the metric tensor in a simple way:

$$R^i_{jkl} = K(\delta^i_k g_{jl} - \delta^i_l g_{jk}) \tag{2.7}$$

where K is the constant Gaussian curvature.

We are particularly interested in those symmetric spaces which have a complex structure. This is a linear endomorphism $J: \mathfrak{m} \to \mathfrak{m}$ satisfying $J^2 = -1$. The vector subspace $\mathfrak{m}$ must have even real dimension. Hermitian symmetric spaces are very special. For this paper, the most useful properties are algebraic:

(i) $\exists A \in \mathfrak{h}$ s.t. $\quad \mathfrak{k} = C_{\mathfrak{g}}(A) = \{B \in \mathfrak{g}: [B, A] = 0\}$

(ii) For a particular scaling of A, $J = \operatorname{ad} A$

(iii) $\exists$ a subset $\theta^+ \subset \phi^+$ of the positive root system s.t.

$$\mathfrak{m} = \operatorname{span}\{e_{\pm\alpha}\}_{\alpha\in\theta^+} \text{ and } \alpha(A) \text{ is constant on } \theta^+.$$

(iv) Following from (iii) $[e_\alpha, e_\beta] = 0$ if $\alpha, \beta \in \theta^+$ or $\alpha, \beta \in \theta^-$. (2.8)

3. The linear problem

Suppose the Lie algebra $\mathfrak{g}$ is Hermitian symmetric (see (2.3) and (2.8)):

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{3.1}$$

with $\mathfrak{k} = C_{\mathfrak{g}}(A)$, $A \in \mathfrak{h}$. As before, $\mathfrak{m}$ is identified with the tangent space $T_{p_0}(G/K)$ of the symmetric space G/K. Choose a basis for which the Cartan subalgebra $\mathfrak{h}$ is represented by diagonal matrices.

Consider the linear equations:

$$\phi_x = (\lambda^2 A + \lambda Q(x, t))\phi \tag{3.2a}$$

$$\phi_t = S(x, t; \lambda)\phi \tag{3.2b}$$

where $Q(x, t) \in \mathfrak{m}$ and $S(x, t; \lambda) \in \mathfrak{g}$. The integrability conditions of (3.2) are:

$$\lambda Q_t = S_x - \lambda[Q, S] - \lambda^2[A, S]. \tag{3.3}$$

The matrix S may be decomposed in terms of (3.1):

$$S(x, t; \lambda) = S_k(x, t; \lambda) + S_m(x, t; \lambda). \tag{3.4}$$

The integrability conditions (3.3) then decouple to give:

$$\begin{aligned} \lambda Q_t &= S_{mx} - \lambda[Q, S_k] - \lambda^2[A, S_m] \\ S_{kx} &= \lambda[Q, S_m] \end{aligned} \tag{3.5}$$

where we have used $[A, S_k]=0$ since $S_k \in C_{\mathfrak{g}}(A)$. It is important to realise that the integrability conditions have been rendered very simple by the special properties of both a symmetric algebra (2.3) and the constant element A. For an nth-order polynomial flow $S=\Sigma_{j=0}^{2n} S^{(j)}\lambda^j$ and (3.5) decouples further into a system of equations for $S_k^{(j)}$ and $S_m^{(j)}$. These equations can be solved recursively, although the calculation is not quite so simple as for the NLS equation. The second-order flow is particularly easy to construct.

3.1. *The second-order flow* $S=\Sigma_{j=0}^{4} S^{(j)}\lambda^j$

It is convenient to make the following assumption:

$$S_m^{(4)} = S_k^{(3)} = S_m^{(2)} = S_k^{(1)} = S_m^{(0)} = S_k^{(0)} = 0, \qquad S_k^{(4)} \text{ constant.} \tag{3.6}$$

This assumption is in accord with the well known case (1.3b). Furthermore it is possible to invoke both the scaling and phase symmetry of (3.2a) to prove the necessity of (3.6). Sufficiency is proved by the following calculation.

We need to solve

$$[Q, S_k^{(4)}]+[A, S_m^{(3)}]=0 \tag{3.7a}$$

$$[Q, S_m^{(3)}]=0 \tag{3.7b}$$

$$S_{mx}^{(3)}-[Q, S_k^{(2)}]-[A, S_m^{(1)}]=0 \tag{3.7c}$$

$$S_{kx}^{(2)}=[Q, S_m^{(1)}] \tag{3.7d}$$

$$Q_t = S_{mx}^{(1)}. \tag{3.7e}$$

The first four of these are used to construct S while the remaining equation is the evolution equation to be solved by inverse scattering. This evolution equation is locally defined only if the constant matrix $S_k^{(4)}$ is chosen carefully. With $S_k^{(4)}=A$, (3.7a) implies:

$$[A, S_m^{(3)}-Q]=0 \Rightarrow S_m^{(3)}=Q \tag{3.8}$$

so that (3.7b) is identically satisfied. The solution of (3.7c)–(3.7d) is a little tricky. So as not to interrupt the main argument we just give the solution here, relegating the proof to an appendix.

First, recall the decomposition (2.8 iii). This means that there exist functions $q^\alpha(x,t)$ and $p^\alpha(x,t)$ such that:

$$Q=\sum_{\alpha\in\theta^+}(q^\alpha e_\alpha + p^\alpha e_{-\alpha}) \tag{3.9}$$

and that $\alpha(A)$ is a non-zero constant on θ^+; set $\alpha(A)=a\ \forall\alpha\in\theta^+$. Then

$$S_k^{(2)} = -\frac{1}{a}\sum_{\alpha,\beta\in\theta^+} q^\alpha p^\beta [e_\alpha, e_{-\beta}]$$

$$S_m^{(1)} = \frac{1}{a} \sum_{\alpha \in \theta^+} (q_x^\alpha e_\alpha - p_x^\alpha e_{-\alpha})$$

$$- \frac{1}{a^2} \sum_{\beta,\gamma,\delta \in \theta^+} (q^\beta q^\gamma p^\delta [e_\beta, [e_\gamma, e_{-\delta}]] + p^\beta p^\gamma q^\delta [e_{-\beta}, [e_{-\gamma}, e_\delta]]). \tag{3.10}$$

In order to decouple the evolution equations (3.7e), note the following: either $\alpha + \beta - \gamma$ is not a root or $\alpha + \beta - \gamma \in \theta^+ \forall \alpha, \beta, \gamma \in \theta^+$ since $(\alpha + \beta - \gamma)(A) = a$. This implies:

$$\sum_{\alpha \in \theta^+} q_t^\alpha e_\alpha = \left(\frac{1}{a} \sum_{\alpha \in \theta^+} q_x^\alpha e_\alpha - \frac{1}{a^2} \sum_{\beta,\gamma,\delta \in \theta^+} q^\beta q^\gamma p^\delta [e_\beta, [e_\gamma, e_{-\delta}]] \right)_x$$

$$\sum_{\alpha \in \theta^+} p_t^\alpha e_{-\alpha} = -\left(\frac{1}{a} \sum_{\alpha \in \theta^+} p_x^\alpha e_{-\alpha} + \frac{1}{a^2} \sum_{\beta,\gamma,\delta \in \theta^+} p^\beta p^\gamma q^\delta [e_{-\beta}, [e_{-\gamma}, e_\delta]] \right)_x. \tag{3.11}$$

These equations can be decoupled even further by using the definition (2.4) of the Riemann curvature tensor. Using components (2.5) with respect to the basis $\{e_{\pm\alpha}\}_{\alpha \in \theta^+}$:

$$\begin{aligned} a q_t^\alpha &= (q_x^\alpha - a^{-1} R^\alpha_{\beta\gamma-\delta} q^\beta q^\gamma p^\delta)_x \\ a p_t^\alpha &= (-p_x^\alpha - a^{-1} R^{-\alpha}_{-\beta-\gamma\delta} p^\beta p^\gamma q^\delta)_x. \end{aligned} \tag{3.12}$$

These equations have Hamiltonian form:

$$a q_t^\alpha = g^{\alpha-\beta} \partial\, \delta H / \delta p^\beta, \qquad a p_t^\alpha = g^{-\alpha\beta} \partial\, \delta H / \delta q^\beta \tag{3.13}$$

with

$$H = -\tfrac{1}{2}(g_{\alpha-\beta}(q^\alpha p_x^\beta - q_x^\alpha p^\beta) + a^{-1} g_{\rho-\alpha} R^\rho_{\beta\gamma-\delta} p^\alpha q^\beta q^\gamma p^\delta). \tag{3.14}$$

The summation convention has been used here and $g_{\alpha-\beta}$ is the metric tensor (2.6). To derive (3.12) from (3.13) it is necessary to use some of the algebraic identities of the Riemann tensor.

Since in the corresponding Hermitian symmetric space the Riemann tensor has the property:

$$(R^\alpha_{\beta\gamma-\delta})^* = R^{-\alpha}_{-\beta-\gamma\delta} \tag{3.15}$$

we can set $p^\alpha = \pm(q^\alpha)^*$ with $a = i$. The minus and plus signs correspond to the compact and non-compact forms respectively. Also in this case, the metric $g_{\alpha-\beta}$ is Hermitian, making the Hamiltonian operator:

$$\begin{pmatrix} 0 & g^{\alpha-\beta} \\ g^{-\alpha\beta} & 0 \end{pmatrix} \partial, \qquad \alpha, \beta \in \theta^+ \tag{3.16}$$

skew Hermitian.

4. Classification and examples

On page 518 of Helgason's book (1978) there is a table of symmetric spaces. Directly beneath this table those spaces which are Hermitian are listed. Following the abstract discussion of § 3 we now give examples of the linear problem (3.2a) and DNLS equations (3.12) associated with each of the Hermitian symmetric spaces. The matrix

S is easily calculated from the formulae of § 3 and is left to the reader.

A III $\dfrac{\mathrm{SU}(p+q)}{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))}$

This example includes the case of the vector DNLS (1.2*b*). The linear problem is an equation in the Lie algebra $\mathrm{su}(p+q)$, which is the compact real form associated with the root space A_{p+q-1}. When $p=q=2$ we have:

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}_x=\left(\begin{array}{cc|cc}\frac{1}{2}\mathrm{i}\lambda^2 & 0 & \lambda q_1 & \lambda q_2\\ 0 & \frac{1}{2}\mathrm{i}\lambda^2 & \lambda q_4 & \lambda q_3\\ \hline -\lambda q_1^* & -\lambda q_4^* & -\frac{1}{2}\mathrm{i}\lambda^2 & 0\\ -\lambda q_2^* & -\lambda q_3^* & 0 & -\frac{1}{2}\mathrm{i}\lambda^2\end{array}\right)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}. \tag{4.1}$$

The choice of A makes $\alpha(A)=\mathrm{i}$ in the top right-hand block. The first two components of the second-order flow are:

$$\begin{aligned}\mathrm{i}q_{1t}&=q_{1xx}+2\mathrm{i}\Big(\sum_{j\neq 3}q_jq_j^*q_1+q_2q_4q_3^*\Big)_x\\ \mathrm{i}q_{2t}&=q_{2xx}+2\mathrm{i}\Big(\sum_{j\neq 4}q_jq_j^*q_2+q_1q_3q_4^*\Big)_x\end{aligned} \tag{4.2}$$

The second two are generated from (4.2) by the interchange $1\leftrightarrow 3$, $2\leftrightarrow 4$. There are then the four complex conjugate equations.

Remark. The choice of compact real form $\mathrm{su}(p+q)$ corresponds to setting $p_j=-q_j^*$. The noncompact real form $\mathrm{su}(p,q)$ corresponds to $p_j=q_j^*$.

When $p=1$ we are dealing with the usual vector NLS equation and the symmetric space is just complex projective space $\mathbb{C}P^q$. Since this is a space of constant curvature K we use (2.7) to obtain:

$$\mathrm{i}q_t^\alpha=q_{xx}^\alpha+\mathrm{i}\Big(K\Big(\sum_{\beta,\gamma\in\theta^+}g_{\beta-\gamma}q^\beta(q^\gamma)^*\Big)q^\alpha\Big)_x \tag{4.3}$$

and its complex conjugate. The metric is given by the Killing form (2.6) restricted to the symmetric space. However, the Killing form is proportional to the trace form in the fundamental representation. With respect to the root vector basis (which is *not* a coordinate basis) the metric is thus proportional to $\delta_{\beta,-\gamma}$, which gives the usual form of (4.3).

CI $\mathrm{Sp}(n)/\mathrm{U}(n)$

The compact group $\mathrm{Sp}(n)$ (sometimes called $\mathrm{USp}(2n)$) of $2n\times 2n$ matrices which are both symplectic and unitary is associated with the root space C_n. For the simplest of these $n=2$.

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}_x=\left(\begin{array}{cc|cc}\frac{1}{2}\mathrm{i}\lambda^2 & 0 & \lambda q_1 & \lambda q_2\\ 0 & \frac{1}{2}\mathrm{i}\lambda^2 & \lambda q_2 & \lambda q_3\\ \hline -\lambda q_1^* & -\lambda q_2^* & -\frac{1}{2}\mathrm{i}\lambda^2 & 0\\ -\lambda q_2^* & -\lambda q_3^* & 0 & -\frac{1}{2}\mathrm{i}\lambda^2\end{array}\right)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}. \tag{4.4}$$

Notice that this is a reduction of (4.1) with $q_4\equiv q_2$. The DNLS equations for this case are given by (4.2) with the same reduction. In general:

$$\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}\subset\frac{\mathrm{SU}(2n)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))}$$

and corresponds to each of the off-diagonal blocks being symmetric. The non-compact real form $\mathrm{Sp}(n, R)$ corresponds to the choice $p_i = q_i^*$.

DIII $\qquad \mathrm{SO}(2n)/\mathrm{U}(n)$

The orthogonal algebra $\mathrm{so}(2n)$ is the compact real form associated with the root space D_n. The general case is exemplified by the D_4 linear problem.

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{pmatrix}_x=\left(\begin{array}{cccc|cccc}\frac{1}{2}\mathrm{i}\lambda^2 & 0 & 0 & 0 & 0 & \lambda q_1 & \lambda q_3 & \lambda q_6\\ 0 & \frac{1}{2}\mathrm{i}\lambda^2 & 0 & 0 & -\lambda q_1 & 0 & \lambda q_2 & \lambda q_5\\ 0 & 0 & \frac{1}{2}\mathrm{i}\lambda^2 & 0 & -\lambda q_3 & -\lambda q_2 & 0 & \lambda q_4\\ 0 & 0 & 0 & \frac{1}{2}\mathrm{i}\lambda^2 & -\lambda q_6 & -\lambda q_5 & -\lambda q_4 & 0\\ \hline 0 & \lambda q_1^* & \lambda q_3^* & \lambda q_6^* & -\frac{1}{2}\mathrm{i}\lambda^2 & 0 & 0 & 0\\ -\lambda q_1^* & 0 & \lambda q_2^* & \lambda q_5^* & 0 & -\frac{1}{2}\mathrm{i}\lambda^2 & 0 & 0\\ -\lambda q_3^* & -\lambda q_2^* & 0 & \lambda q_4^* & 0 & 0 & -\frac{1}{2}\mathrm{i}\lambda^2 & 0\\ -\lambda q_6^* & -\lambda q_5^* & -\lambda q_4^* & 0 & 0 & 0 & 0 & -\frac{1}{2}\mathrm{i}\lambda^2\end{array}\right)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{pmatrix}. \tag{4.5}$$

There are three basic equations

$$\begin{aligned}\mathrm{i}q_{1t}&=q_{1xx}+2\mathrm{i}\Big(q_1\sum_{j\neq 4}q_jq_j^*+q_4^*(q_3q_5-q_2q_6)\Big)_x\\ \mathrm{i}q_{2t}&=q_{2xx}+2\mathrm{i}\Big(q_2\sum_{j\neq 6}q_jq_j^*+q_6^*(q_3q_5-q_1q_4)\Big)_x\\ \mathrm{i}q_{3t}&=q_{3xx}+2\mathrm{i}\Big(q_3\sum_{j\neq 5}q_jq_j^*+q_5^*(q_1q_4+q_2q_6)\Big)_x .\end{aligned}\tag{4.6}$$

Another three are obtained by making the interchanges $1\leftrightarrow 4, 2\leftrightarrow 6, 3\leftrightarrow 5$. The system is completed by complex conjugation. This is another reduction of the AIII case:

$$\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)}\subset\frac{\mathrm{SU}(2n)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))}$$

this time corresponding to each of the off-diagonal blocks being anti-symmetric.

Notice that $q_4\equiv q_5\equiv q_6\equiv 0$ is a consistent reduction. This corresponds to taking the subsystem D_3 of D_4. Furthermore, this reduction is identical to the 3-component *vector* DNLS equation. This corresponds to the isomorphism $\mathrm{D}_3\simeq\mathrm{A}_3$, leading to

$$\frac{\mathrm{SO}(6)}{\mathrm{U}(3)}\simeq\frac{\mathrm{SU}(4)}{\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(3))}$$

BD1 $\qquad \dfrac{\mathrm{SO}(p+q)}{\mathrm{SO}(p)\times\mathrm{SO}(q)},\qquad p=2$

This symmetric space is only Hermitian when $p=2$. In general $\mathrm{so}(p)\oplus\mathrm{so}(q)$ has no centre. When $p=2$ the $\mathrm{so}(2)$ subalgebra is the centre. Depending upon whether

q is odd or even this symmetric space is associated with either $B_{(q+1)/2}$ or $D_{(q+2)/2}$. The simplest non-trivial example is associated with D_3. Even though $D_3 \simeq A_3$, $SO(2)\times SO(4) \not\simeq S(U(1)\times U(3))$. Indeed, this example has four independent components q_i, not just three. The eigenvalue problem is possibly best understood in terms of the skew symmetric matrix representations (Helgason 1978). However, the present calculation is more easily performed in the representation given by Humphreys (1972). The eigenvalue problem is

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{pmatrix}_x = \left(\begin{array}{ccc|ccc} \mathrm{i}\lambda^2 & \lambda q_1 & \lambda q_2 & 0 & \lambda q_3 & \lambda q_4 \\ -\lambda q_1^* & 0 & 0 & -\lambda q_3 & 0 & 0 \\ -\lambda q_2^* & 0 & 0 & -\lambda q_4 & 0 & 0 \\ \hline 0 & \lambda q_3^* & \lambda q_4^* & -\mathrm{i}\lambda^2 & \lambda q_1^* & \lambda q_2^* \\ -\lambda q_3^* & 0 & 0 & -\lambda q_1 & 0 & 0 \\ -\lambda q_4^* & 0 & 0 & -\lambda q_2 & 0 & 0 \end{array}\right) \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{pmatrix}. \tag{4.7}$$

There are two basic equations

$$\begin{aligned} \mathrm{i}q_{1t} &= q_{1xx} + 2\mathrm{i}\Big(q_1 \sum_{j\neq 3} q_j q_j^* - q_2 q_4 q_3^*\Big)_x \\ \mathrm{i}q_{2t} &= q_{2xx} + 2\mathrm{i}\Big(q_2 \sum_{j\neq 4} q_j q_j^* - q_1 q_3 q_4^*\Big)_x \end{aligned} \tag{4.8}$$

Two more equations are obtained by the interchange $1\leftrightarrow 3$, $2\leftrightarrow 4$. There are then the complex conjugates of these four.

4.1. Exceptional algebras

All the examples so far given have been associated with the classical Lie algebras. There are two Hermitian symmetric spaces EIII and EVII, associated with the exceptional E-series. They possess respectively 16 and 27 independent complex potentials, q_i, which would satisfy a corresponding system of generalised DNLS equations. These examples are left as an exercise for the reader.

5. Conclusions

This paper and its precursor (Fordy and Kulish 1983) have generalised the well known vector DNLS and NLS equations (1.2) to similar equations associated with arbitrary Hermitian symmetric spaces. The mixed single component NLS-DNLS equation of Wadati *et al* (1979a) can be similarly generalised in an obvious fashion. The slightly modified derivative nonlinear Schrödinger equation of Chen *et al* (1979):

$$\mathrm{i}q_t = q_{xx} + 2\mathrm{i}qq^* q_x \tag{5.1}$$

has a linear problem (Dodd and Fordy 1983a, b) which is gauge equivalent to (1.3b) with $n=1$. A multicomponent generalisation of (5.1) exists for each of the symmetric spaces encountered in this paper. The linear problem (3.2) is gauge transformed with an element of the subgroup K.

However, the more exotic generalisation of Wadati *et al* (1979b) seems to be restricted to the single component case. This will also be true of some of the other single component, complex equations found in Dodd and Fordy (1983b).

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Appendix

In this appendix we indicate the proof of (3.10). We need to solve (3.7*c*)–(3.7*d*):

$$[A, S_m^{(1)}] = Q_x - [Q, S_k^{(2)}] \tag{A1a}$$

$$S_{kx}^{(2)} = [Q, S_m^{(1)}] \tag{A1b}$$

Since the subalgebra $\mathfrak{k}$ is spanned by $[e_\beta, e_{-\gamma}]$, β, $\gamma \in \theta^+$, there exist functions $S_{k\beta\gamma}^{(2)}$ such that:

$$S_k^{(2)} = S_{k\beta\gamma}^{(2)}[e_\beta, e_{-\gamma}]. \tag{A2}$$

Here and elsewhere in this appendix the summation convention is used. Therefore,

$$[Q, S_k^{(2)}] = S_{k\beta\gamma}^{(2)} q^\alpha [e_\alpha, [e_\beta, e_{-\gamma}]] + S_{k\beta\gamma}^{(2)} p^\alpha [e_{-\alpha}, [e_\beta, e_{-\gamma}]] \tag{A3}$$

so that

$$S_m^{(1)} = a^{-1}(q_x^\alpha e_\alpha - p_x^\alpha e_{-\alpha} + S_{k\beta\gamma}^{(2)}(q^\alpha[e_\alpha, e_\beta, e_{-\gamma}]] - p^\alpha[e_{-\alpha}, [e_\beta, e_{-\gamma}]])) \tag{A4}$$

For $S_{k\beta\gamma}^{(2)}$ to be locally defined, the right-hand side of (A1*b*) must be an exact derivative. There are two parts to this right-hand side: first

$$a^{-1}[q^\delta e_\delta + p^\delta e_{-\delta}, q_x^\alpha e_\alpha - p_x^\alpha e_{-\alpha}] = -a^{-1}(q^\beta p^\gamma)_x[e_\beta, e_{-\gamma}]. \tag{A5}$$

The second part is

$$a^{-1} S_{k\beta\gamma}^{(2)}[q^\delta e_\delta + p^\delta e_{-\delta}, q^\alpha[e_\alpha, e_\beta, e_{-\gamma}]] - p^\alpha[e_{-\alpha}, [e_\beta, e_{-\gamma}]]] \tag{A6}$$

(A6) requires some manipulation, using Jacobi's identities and $[e_\delta, [e_\alpha, [e_\beta, e_{-\gamma}]]] = 0$. The latter follows from (2.8iv). Expression (A6) reduces to

$$a^{-1} S_{k\beta\gamma}^{(2)} q^\delta p^\alpha [[e_\beta, e_{-\gamma}], [e_\delta, e_{-\alpha}]] \tag{A7}$$

which vanishes provided $S_{k\beta\gamma}^{(2)} \propto q^\beta p^\gamma$ (remember the summation convention). However, this proportionality is very reasonable, since it is implied by the scaling and phase symmetries. Furthermore, when (A7) vanishes (A1*b*) and (A5) imply:

$$S_k^{(2)} = -a^{-1} q^\beta p^\gamma [e_\beta, e_{-\gamma}] \tag{A8}$$

which is the solution (3.10).

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